COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **13** (4) (1993) 467–476

# SOME EXCLUDED-MINOR THEOREMS FOR A CLASS OF POLYMATROIDS

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*Received July 6, 1990*
*Revised September 27, 1991*

Problems involving representability are among the most frequently studied of all the problems in matroid theory. This paper considers the corresponding class of problems for polymatroids. A polymatroid $h$ on the set $S$ is representable over a free matroid or is Boolean if there is a map $\phi$ from $S$ into the set of subsets of a set $V$ which preserves rank, that is for all subsets $A$ of $S$, $h(A) = \left|\bigcup_{a\in A}\phi(a)\right|$. The class of Boolean polymatroids is minor-closed and in this paper we investigate the excluded minors of this class. In particular, we determine all such Boolean excluded minors that are 2-polymatroids.

## 1. Introduction

Let $S$ be a finite set. A *polymatroid on* $S$ is an integer-valued set function $h:2^S \to Z$ which is *normalized* ($h(\emptyset)=0$), is *increasing* ($h(A) \geq h(B)$ whenever $A$ and $B$ are subsets of $S$ with $A \supseteq B$) and is *submodular* ($h(A)+h(B) \geq h(A\cup B)+h(A\cap B)$ for all subsets $A$ and $B$ of $S$). The set $S$ is called the *ground set* of $h$. If $k$ is a positive integer, then a *k-polymatroid* is a polymatroid whose value on singletons never exceeds $k$. It is well known, and easily seen, that the 1-polymatroids on $S$ are precisely the rank functions of matroids on $S$. By analogy with matroids, the value of $h(S)$ is called the *rank* of $h$.

The terminology used above essentially agrees with that of Lovász [5,6] and Lovász and Plummer [7], but note that polymatroids have also been called hypermatroids (see for example, [2,4]). The reader is warned that the term "polymatroid" is often used to refer to a polyhedron which can be associated with the set function $h$. We shall not follow this practice.

Let $M$ be a matroid having ground set $E$ and let $h$ be a polymatroid on $S$. A function $\phi : S \to 2^E$ is a *representation* of $h$ in $M$ if $h(A) = r_M(\cup_{a\in A}\phi(a))$ for all subsets $A$ of $S$. If there exists a representation of $h$ in $M$, then $h$ is said to be *representable* over $M$. If $h$ is representable over $PG(r-1,q)$ for some natural

AMS subject classification code (1991): 05 B 35

* This research was partially supported by a grant from the Louisiana Education Quality Support Fund Through the Board of Regents

** This research was supported by a grant from the Commonwealth of Australia through the Australian Research Council

number $r$, then we shall say that $h$ is representable over $GF(q)$. Clearly this terminology agrees with the usual matroid terminology when $h$ is the rank function of a matroid. It is well known that every polymatroid is representable over some matroid. A polymatroid is *Boolean* if it is representable over some free matroid. In other words, the polymatroid $h$ on $S$ is Boolean if there exists a finite set $V$ and a function $\phi: S \to 2^V$ such that $h(A) = |\cup_{a\in A}\phi(a)|$ for all subsets $A$ of $V$.

The class of Boolean 2-polymatroids is of particular interest, as these structures are essentially graphs. For a graph $G$, let $h_G$ be the set function whose value on all subsets $A$ of $E(G)$ is the number of vertices incident with edges in $A$. Then a 2-polymatroid is Boolean if and only if it is equal to $h_G$ for some graph $G$. Moreover it is easily seen that this construction gives a one-to-one correspondence between the class of Boolean 2-polymatroids and the class of graphs with no isolated vertices.

This paper is concerned with the problem of determining when a given polymatroid is Boolean; in particular, we are interested in excluded minors of Boolean polymatroids. For a definition of "excluded minor" see Section 2; the notion is a straightforward generalization of that for matroids.

In Section 2, Boolean excluded minors are characterised in terms of an evaluation of a function which can naturally be associated with a polymatroid. In Section 3 it is shown that the ground set of a $k$-polymatroid that is a Boolean excluded minor has cardinality at most $k+2$. It follows that for any positive integer $k$, there are only a finite number of Boolean excluded minors that are $k$-polymatroids. Section 3 also presents an example which shows that, in general, there are infinitely many Boolean excluded minors. In Section 4, a complete list of Boolean excluded minors which are 2-polymatroids is given. Via the correspondence noted above, this result can be somewhat loosely interpreted as an excluded minor characterisation of the class of graphs within the class of 2-polymatroids.

## 2. Some properties of Boolean excluded minors

Let $h$ be a polymatroid on $S$ and let $A$ be a subset of $S$. The *deletion* of $A$ from $h$, denoted $h\backslash A$, is the restriction of $h$ to the power set of $S-A$; that is, $h\backslash A(B) = h(B)$ for all subsets $B$ of $S-A$. By a slight abuse of language, we frequently call $h\backslash A$ the *restriction* of $h$ to $S-A$ and denote it by $h|(S-A)$. The *contraction* of $A$ from $h$, denoted $h/A$, is the function defined by $h/A(B) = h(A\cup B) - h(A)$ for all subsets $B$ of $S-A$.

The above definitions extend familiar ones from matroid theory in an obvious way. It is easily seen that deletion and contraction commute, both with themselves and with each other. A *minor* of $h$ is a set function which can be obtained from $h$ by a sequence of deletions and contractions. Equivalently, a minor of $h$ is a set function of the form $h\backslash X/Y$ for some pair of disjoint subsets $X$ and $Y$ of $S$. It is routinely verified that minors of $k$-polymatroids are $k$-polymatroids.

The definition given above of deletion is uncontentious; the definition of contraction perhaps needs some justification. This is provided by the following proposition whose straightforward proof is omitted.

**Proposition 2.1.** *Let $M$ be a matroid on $E$, $h$ be a polymatroid on $S$, and $\phi$ be a representation of $h$ in $M$. If $s \in S$, then the function $\phi'$ defined, for all elements of $S-\{s\}$, by $\phi'(x)=\phi(x)-\phi(s)$ is a representation of $h/s$ in $M/\phi(s)$.* ∎

Let $\mathcal{A}$ be a class of polymatroids which is closed under the taking of minors. A polymatroid not isomorphic to any member of $\mathcal{A}$ but having the property that each of its proper minors is isomorphic to a member of $\mathcal{A}$ is an *excluded minor* of $\mathcal{A}$. An excluded minor for the class of Boolean polymatroids will be called a *Boolean excluded minor.*

If $h$ is a polymatroid on $S$ and $A$ and $B$ are subsets of $S$, then we define $D_h(A,B)$ by

$$D_h(A,B) = \sum_{C \subseteq S-(A\cap B)} (-1)^{|C|}\{h(A\cup C)+h(B\cup C)-h(A\cup B\cup C)-h((A\cap B)\cup C)\}.$$

Boolean polymatroids can then be characterised by the following:

**Proposition 2.2** (Helgason [2, Theorem 1.11]). *The polymatroid $h$ on $S$ is Boolean if and only if $D_h(A,B) \geq 0$ for all subsets $A$ and $B$ of $S$.* ∎

To show that a polymatroid is not Boolean, all that is required is a pair of subsets $A$ and $B$ for which $D_h(A,B)<0$. But, if $A$ and $B$ partition $S$, then, relative to the size of the ground set of $h$, $D_h(A,B)$ is evaluated by a sum with exponentially many terms. This means that one cannot say that Boolean polymatroids are well-characterised by Proposition 2.2. Nonetheless, Proposition 2.2 is valuable.

The *Boolean characteristic function* $B(h;\lambda)$ of the polymatroid $h$ on $S$ is defined by

$$B(h;\lambda) = \sum_{A \subseteq S} (-1)^{|A|}(h(S) - h(A))^{\lambda}.$$

The following proposition indicates the combinatorial significance of this function. It is noted in [8]. (See also [3].)

**Proposition 2.3.** *Let $h$ be a Boolean polymatroid on $S$ and let $\phi$ be a representation of $h$ in the free matroid on $V$. Then for any integer $k$, $B(h;k) \geq 0$, and the least positive integer $k$ for which $B(h;k)$ is positive is equal to the least positive integer $k$ for which there is a subset $W$ of $V$ with $|W|=|V|-k$ such that $W$ contains no member of $\{\phi(s): s \in S\}$.* ∎

Using appropriate polymatroids associated with graphs, Boolean characteristic functions can be used — via Proposition 2.3 — to compute the size of a maximum clique, the size of a maximum independent set of vertices, and the size of a maximum matching of a graph [8]. In what follows we only need to evaluate $B(h;1)$, but it is worth noting the connection between the theory here and that in [3] and [8]. We summarise relevant information about $B(h;1)$ in the following proposition, noting that the proof repeatedly uses the elementary fact that if $X$ is a non-empty finite set, then $\sum_{A \subseteq X} (-1)^{|A|}=0$.

**Proposition 2.4.** *Let $h$ be a polymatroid on $S$.*

(i) $B(h;1) = -\sum_{A \subseteq S}(-1)^{|A|}h(A)$.

(ii) *If $x$ is an element of $S$ and $|S| \geq 2$, then* $B(h;1) = B(h\setminus x;1) - B(h/x;1)$.

(iii) *If $h$ is Boolean, then* $B(h;1) \geq 0$.

**Proof.**

(i)
$$\begin{aligned} B(h;1) &= \sum_{A \subseteq S}(-1)^{|A|}(h(S) - h(A)) \\ &= \sum_{A \subseteq S}(-1)^{|A|}h(S) - \sum_{A \subseteq S}(-1)^{|A|}h(A) \\ &= -\sum_{A \subseteq S}(-1)^{|A|}h(A). \end{aligned}$$

(ii) $B(h\setminus x;1) - B(h/x;1)$
$$\begin{aligned} &= -\sum_{A \subseteq S-x}(-1)^{|A|}h(A) + \sum_{A \subseteq S-x}(-1)^{|A|}(h(A \cup x) - h(\{x\})) \\ &= -\sum_{A \subseteq S-x}(-1)^{|A|}h(A) + \sum_{A \subseteq S-x}(-1)^{|A|}h(A \cup x) - \sum_{A \subseteq S-x}(-1)^{|A|}h(\{x\}). \end{aligned}$$

The last term is zero since $S-x$ is non-empty. Thus
$$B(h\setminus x;1) - B(h/x;1) = -\sum_{A \subseteq S}(-1)^{|A|}h(A) = B(h;1).$$

(iii) This follows immediately from Proposition 2.3. ∎

Note that, in general, $B(h;\lambda) = B(h\setminus x;\lambda) - B(h/x;\lambda)$ if and only if $h(S) = h(S-x)$. The following theorem is the main result of this section.

**Theorem 2.5.** *The polymatroid $h$ is a Boolean excluded minor if and only if $B(h;1) < 0$, and, whenever $h'$ is a proper minor of $h$, $B(h';1) \geq 0$.*

We first prove two lemmas.

**Lemma 2.6.** *Let $h$ be a polymatroid on $S$ and $A$ and $B$ be subsets of $S$ such that $D_h(A,B) < 0$. Then $h/(A \cap B)$ is not Boolean.*

**Proof.**
$$\begin{aligned} &D_{h/(A\cap B)}(A-B, B-A) \\ &= \sum_{C \subseteq S-(A\cap B)}(-1)^{|C|}\{h/(A\cap B)((A-B)\cup C) + h/(A\cap B)((B-A)\cup C) \\ &\qquad - h/(A\cap B)((A-B)\cup(B-A)\cup C) - h/(A\cap B)(C)\} \\ &= \sum_{C \subseteq S-(A\cap B)}(-1)^{|C|}\{h(A\cup C) + h(B\cup C) - h(A\cup B\cup C) - h((A\cap B)\cup C)\} \\ &= D_h(A,B). \end{aligned}$$

But $D_h(A,B)<0$. Therefore, by Proposition 2.2, $h/(A\cap B)$ is not Boolean. ∎

**Lemma 2.7.** *If $h$ is a polymatroid on $S$, and $A$ and $B$ are non-empty disjoint subsets of $S$, then $D_h(A,B)=B(h;1)$.*

**Proof.**

$$D_h(A,B) = \sum_{C\subseteq S}(-1)^{|C|}\{h(A\cup C)+h(B\cup C)-h(A\cup B\cup C)-h(C)\} =$$

$$\sum_{C\subseteq S}(-1)^{|C|}h(A\cup C)+\sum_{C\subseteq S}(-1)^{|C|}h(B\cup C)-\sum_{C\subseteq S}(-1)^{|C|}h(A\cup B\cup C)+B(h;1).$$

Now

$$\sum_{C\subseteq S}(-1)^{|C|}h(A\cup C) = \sum_{X\subseteq A, Y\subseteq S-A}(-1)^{|X|+|Y|}h(A\cup Y)$$
$$= \left(\sum_{X\subseteq A}(-1)^{|X|}\right)\left(\sum_{Y\subseteq S-A}(-1)^{|Y|}h(A\cup Y)\right).$$

But $A$ is non-empty, so $\sum_{X\subseteq A}(-1)^{|X|}=0$, and hence $\sum_{C\subseteq S}(-1)^{|C|}h(A\cup C)=0$. Similarly $\sum_{C\subseteq S}(-1)^{|C|}h(B\cup C)=0$, and $\sum_{C\subseteq S}(-1)^{|C|}h(A\cup B\cup C)=0$, so the result follows. ∎

**Proof of Theorem 2.5.** Assume that $h$ is a Boolean excluded minor. Then there exist subsets $A$ and $B$ of the ground set $S$ of $h$ such that $D_h(A,B)<0$. It follows from Lemma 2.6 that $A$ and $B$ are disjoint subsets of $S$ (otherwise $h$ would have a proper minor which is not Boolean). Clearly both $A$ and $B$ are non-empty. Therefore, by Lemma 2.7, $D_h(A,B)=B(h;1)$, so that $B(h;1)<0$. Let $h'$ be a proper minor of $h$. If the ground set of $h'$ has at most one element, then $B(h';1)\geq 0$. Hence we may suppose that the ground set of $h'$ has at least two elements. Then this ground set has disjoint non-empty subsets $A$ and $B$. Moreover, as $h'$ is Boolean, Proposition 2.2 implies that $D'_h(A,B)\geq 0$. But by Lemma 2.7, $D_{h'}(A,B)=B(h';1)$ and hence $B(h';1)\geq 0$.

The proof of the converse is just as routine and is omitted. ∎

Theorem 2.5 suggests a conjecture for representability of polymatroids over finite fields. The *characteristic polynomial* of the polymatroid $h$ on $S$, denoted $P(h;\lambda)$, is defined by

$$P(h;\lambda) = \sum_{A\subseteq S}(-1)^{|A|}\lambda^{h(S)-h(A)}$$

This is an immediate generalisation of the definition for matroids. It is shown in [8] that characteristic polynomials provide essentially the same information for representations of polymatroids over finite fields as Boolean characteristic functions provide for representations over free matroids. One is then led to ask if it is true that a polymatroid $h$ is an excluded minor for representability over $GF(q)$ if and only if $P(h;q)<0$, and $P(h';q)\geq 0$ whenever $h'$ is a proper minor of $h$.

A negative answer can immediately be given to the question for $GF(q)$ for all $q\geq 3$ even when the polymatroid $h$ is a matroid. The matroid $U_{q,q+2}$ is an excluded minor for representability over $GF(q)$, but it is not difficult to show that

$$P(U_{q,q+2};\lambda) = (\lambda-1)^q - 2(\lambda-1)^{q-1} + 3(\lambda-1)^{q-2} - \cdots + (-1)^{q-3}q(\lambda-1).$$

Thus $P(U_{q,q+2};q)>0$. For $GF(2)$, the answer is not so immediate. Since $U_{2,4}$ is the only matroid that is an excluded minor for matroids representable over $GF(2)$, one can answer the question in the affirmative for the special case when the polymatroid $h$ is a matroid. However, as the following example shows, for arbitrary $h$, the answer to the question is again negative.

Let $v$ be the polymatroid on the set $\{a,b,c,d\}$ defined as follows:

$$v(A) = \begin{cases} 2, & \text{if } |A| = 1; \\ 3, & \text{if } |A| = 2 \text{ and } A \neq \{a,b\}; \\ 4, & \text{if } A = \{a,b\} \text{ or } |A| > 2. \end{cases}$$

Geometrically $v$ represents four appropriately chosen lines of a Vámos matroid: one could call $v$ the Vámos 2-polymatroid. It is readily verified — the argument is the same as that for the Vámos matroid — that $h$ is not representable over any field, and a routine check shows that all proper minors of $v$ are representable over $GF(2)$, so that $v$ is an excluded minor for representability over $GF(2)$. But $P(v;\lambda)=\lambda^4-4\lambda^2+5\lambda-2$, so that $P(v;2)>0$. This polymatroid is also discussed in [5].

## 3. Bounds on the size of Boolean excluded minors

Let $k$ be a positive integer and $S$ be a set with $k+2$ elements. Let $b_k$ be the polymatroid on $S$ defined, for all non-empty subsets $A$ of $S$, by

$$b_k(A) = \begin{cases} k, & \text{if } |A| = 1 \\ k+1, & \text{otherwise.} \end{cases}$$

Now $B(b_k;1)=-1$, so that $b_k$ is not Boolean. Also, if $s\in S$, then $b_k\setminus s$ is clearly representable as the collection of hyperplanes of a $(k+1)$-simplex, and $b_k/s$ is the matroid $U_{1,k+1}$ so that both $h\setminus s$ and $h/s$ are Boolean. Therefore $b_k$ is a Boolean excluded minor.

It follows from the above example that there are an infinite number of Boolean excluded minors. The main result of this section is the following theorem from which it follows that, for each positive integer $k$, there are only finitely many Boolean excluded minors that are $k$-polymatroids.

**Theorem 3.1.** *Let the $k$-polymatroid $h$ on $S$ be a Boolean excluded minor. Then $3\leq|S|\leq k+2$. Moreover, if $|S|=k+2$, then $h=b_k$.*

We shall need some preliminary results. Let $\phi$ represent the Boolean polymatroid $h$ on $S$ in the free matroid on $V$. Associated with this representation is a simple bipartite graph with vertex set $S \cup V$. An edge joins $s \in S$ to $v \in V$ if and only if $v \in \phi(s)$. We call this graph a *bipartite representation* of $h$. No new information is gained by using bipartite representations, but they do provide a useful visual aid. The proof of the following lemma is straightforward and is omitted. If $X$ is a subset of the vertex set of a graph $G$, then $G-X$ denotes the graph that is obtained from $G$ by deleting the vertices in $X$.

**Lemma 3.2.** *Let $\phi$ represent the Boolean polymatroid $h$ on $S$ in the free matroid on $V$, and let $B$ be the associated bipartite representation of $h$. If $s$ is an element of $S$, then $B-s$ is a bipartite representation of $h\backslash s$, and $B-(\{s\}\cup\phi(s))$ is a bipartite representation of $h/s$.* ∎

A Boolean polymatroid is *affine* if $B(h,1) > 0$. The following lemma is an immediate consequence of Proposition 2.3.

**Lemma 3.3.** *Suppose that a Boolean polymatroid $h$ on $S$ is represented in the free matroid on $V$. Then $h$ is affine if and only if, in the associated bipartite representation of $h$, there is an element $v$ in $V$ that is adjacent to every element of $S$.* ∎

**Lemma 3.4.** *If the polymatroid $h$ on $S$ is a Boolean excluded minor and $x$ is in $S$, then $h/x$ is affine.*

**Proof.** It follows from Theorem 2.5 that $B(h;1) < 0$ and $B(h\backslash x;1) \geq 0$. By Proposition 2.4 (ii), $B(h;1) = B(h\backslash x;1) - B(h/x;1)$. Hence $B(h/x;1) > 0$, and it follows, since $h/x$ is Boolean, that $h/x$ is affine. ∎

**Proof of Theorem 3.1.** It is easy to show that if $|S| \leq 2$, then $h$ is Boolean. Thus $|S| \geq 3$. Establishing the upper bound on $|S|$ will require more effort. Let $x$ be in $S$, and consider $h\backslash x$. Since $h\backslash x$ is Boolean, it has a bipartite representation. Assume that $B$ is such a representation and that the vertex set of $B$ has parts $S-x$ and $V$. Let $z$ be any element of $S-x$. By Lemma 3.4, $h/z$ is affine, and it then follows from Lemmas 3.2 and 3.3 that $h/z\backslash x$ is affine. But $h/z\backslash x = h\backslash x/z$, so that $h\backslash x/z$ is affine.

Now let $y$ be a fixed element of $S-x$ and assume that $S-\{x,y\} = \{x_1, x_2, \ldots, x_m\}$. It follows from Lemma 3.2 that if $1 \leq i \leq m$, then $h\backslash x/x_i$ can be represented by a subgraph $B_i$ of $B$ obtained in the way prescribed by Lemma 3.2. Since $h\backslash x/x_i$ is affine, it follows from Lemma 3.3 that there exists a vertex $v_i$ of $B_i$ which is adjacent to every element of $S\backslash\{x, x_i\}$. In particular, we may conclude that, in $B$, there exists a vertex $v_i$ of $B$ which is adjacent to every element of $S\backslash\{x, x_i\}$. Now there cannot exist distinct integers $i$ and $j$ such that $v_i = v_j$, for $v_i$ is a vertex of $B_i$, but, by Lemma 3.2, $v_j$ is not. Therefore $|\{v_1, v_2, \ldots, v_m\}| = m$. But it follows from the above that $\{y, v_i\}$ is an edge of $B$ for $1 \leq i \leq m$. Therefore, $y$ has degree at least $m$ in $B$. But it is easily seen that since $h\backslash x$ is a $k$-polymatroid, the degree in $B$ of any vertex in $S-x$ cannot exceed $k$. Thus $m \leq k$, and we conclude that $|S| \leq k+2$. Moreover, it is straightforward to extend the above argument to show that if $|S| = k+2$, then $h = b_k$. ∎

One can also bound the rank of Boolean excluded minors that are $k$-polymatroids:

**Proposition 3.5.** *Let the $k$-polymatroid $h$ on $S$ be a Boolean excluded minor. Then* $h(S) \leq \lfloor \frac{1}{4}k(k+2) \rfloor$. ∎

We have omitted the straightforward argument that proves Proposition 3.5 since we conjecture that the order of the bound given therein can be improved:

**Conjecture 3.6.** *If the $k$-polymatroid $h$ on $S$ is a Boolean excluded minor, then* $h(S) \leq 3k-1$.

The following example shows that the bound of Conjecture 3.6 is attained. Let $S = \{a,b,c\}$ and let $k$ be a positive integer. Define the polymatroid $h$ on $S$ by $h(\emptyset)=0$, $h(\{a\})=h(\{b\})=h(\{c\})=k$, $h(\{a,b\})=h(\{a,c\})=h(\{b,c\})=2k$ and $h(\{a,b,c\})=3k-1$. The routine verification that $h$ is a Boolean excluded minor is left to the reader.

The $k$-polymatroid $h$ is *strict* if its ground set contains an element $x$ with the property that $h(\{x\}) = k$. The polymatroid $b_k$ is an example of a strict $k$-polymatroid of rank $k+1$ which is a Boolean excluded minor. It is clear that no strict $k$-polymatroid of lower rank is a Boolean excluded minor.

While the task of finding all Boolean excluded minors which are $k$-polymatroids is clearly finite for any positive integer $k$, we know of no systematic way of finding them. Despite this one can simplify the task somewhat. Propositions 3.7 and 3.8 below do this. We also include these results as an illustration of how concepts in matroid theory have natural counterparts in polymatroid theory.

The element $a$ of the ground set of the polymatroid $h$ is a *loop* if $h(\{a\})=0$. The element $x$ is *dominated* by the element $y$ if $x$ is a loop of $h/y$. Equivalently, $x$ is dominated by $y$ if $h(\{x,y\})=h(\{y\})$. The polymatroid $h$ is *simple* if no element of its ground set dominates any other. The notion of a simple polymatroid generalises that of a simple matroid.

**Proposition 3.7.** *Boolean excluded minors are simple.*

**Proof.** Assume that $h$ is a non-simple Boolean excluded minor. Then there exists an element $y$ of the ground set of $h$ with the property that $h/y$ has a loop $x$. Now $x$ has no neighbours in the bipartite representation of $h/y$. Hence, by Lemma 3.3, $h/y$ is not affine. But this contradicts Lemma 3.3. ∎

A polymatroid $h$ on $S$ is *connected* [1] if there is no proper non-empty subset $X$ of $S$ such that $h(X)+h(S-X)=h(S)$.

**Proposition 3.8.** *Boolean excluded minors are connected.*

**Proof.** Assume that $h$ is a disconnected Boolean excluded minor. Then, for some proper non-empty subset $X$ of $S$, $h(X)+h(S-X)=h(S)$. It is straightforward to show that, for all $A \subseteq S$, $h(A)=h(A \cap X)+h(A \cap (S-X))$. Thus

$$B(h;1) = \sum_{A \subseteq S} (-1)^{|A|} h(A) = \sum_{A_1 \subseteq X, A_2 \subseteq S-X} (-1)^{|A_1|+|A_2|} (h(A_1) + h(A_2))$$

$$= \left(\sum_{A_1\subseteq X}(-1)^{|A_1|}h(A_1)\right)\left(\sum_{A_2\subseteq S-X}(-1)^{|A_2|}\right) + \left(\sum_{A_2\subseteq S-X}(-1)^{|A_2|}h(A_2)\right)\left(\sum_{A_1\subseteq X}(-1)^{|A_1|}\right) = 0.$$

This contradicts Theorem 2.5. ∎

## 4. Excluded minors for Boolean 2-polymatroids

First consider matroids, that is, 1-polymatroids. It follows from the results in Section 3 that a matroid is a Boolean excluded minor if and only if it is simple, has rank 2, and has a ground set with three elements. Clearly the only such matroid is $U_{2,3}$.

| Excluded Minor | Comments |
|---|---|
| $h_1 : h_1(\{a\}) = h_1(\{b\}) = h_1(\{c\}) = 1, h_1(\{a,b\}) = h_1(\{a,c\}) = h_1(\{b,c\}) = h_1(\{a,b,c\}) = 2$ | The matroid $U_{2,3}$; $h_1 = b_1$. |
| $h_2 : h_2(\{a\}) = 2, h_2(\{b\}) = h_2(\{c\}) = 1, h_2(\{b,c\}) = 2, h_2(\{a,b\}) = h_2(\{a,c\}) = h_2(\{a,b,c\}) = 3$ | A line and two points freely placed in rank 3. |
| $h_3 : h_3(\{a\}) = h_3(\{b\}) = 2, h_3(\{c\}) = 1, h_3(\{a,b\}) = h_3(\{a,c\}) = h_3(\{b,c\}) = h_3(\{a,b,c\}) = 3$ | Two lines and a point freely placed in rank 3. |
| $h_4 : h_4(\{a\}) = h_4(\{b\}) = h_4(\{c\}) = 2, h_4(\{a,b\}) = 3, h_4(\{a,c\}) = h_4(\{b,c\}) = h_4(\{a,b,c\}) = 4$ | |
| $h_5 : h_5(\{a\}) = h_5(\{b\}) = h_5(\{c\}) = 2, h_5(\{a,b\}) = h_5(\{a,c\}) = h_5(\{b,c\}) = h_5(\{a,b,c\}) = 4$ | Three lines freely placed in rank 4. |
| $h_6 : h_6(\{a\}) = h_6(\{b\}) = 2, h_6(\{c\}) = 1, h_6(\{a,c\}) = h_6(\{b,c\}) = 3, h_6(\{a,b\}) = h_6(\{a,b,c\}) = 4$ | Two lines and a point freely placed in rank 4. |
| $h_7 : h_7(\{a\}) = h_7(\{b\}) = h_7(\{c\}) = 2, h_7(\{a,b\}) = h_7(\{b,c\}) = h_7(\{a,c\}) = 4, h_7(\{a,b,c\}) = 5$ | Three lines freely placed in rank 5. |
| $h_8$: a polymatroid on a 4-element ground set which takes the value 2 on singletons and 3 on all other nonempty subsets. | Four lines freely placed in rank 3; $h_8 = b_2$ |

*Table 4.1.* Excluded minors for the class of Boolean 2-polymatroids.

Now consider 2-polymatroids. It follows from Theorem 3.1 that the ground set of a Boolean excluded minor that is a 2-polymatroid has cardinality at most four. Using this fact, and other results from earlier sections, a routine check enables one to find all such excluded minors. These are listed in Table 4.1; the details of the check are omitted since they are uninteresting.

It was noted in the introduction that a Boolean 2-polymatroid $h_G$ can be defined on all subsets of the edge set of a graph $G$ by taking $h_G(A)$ to be the cardinality of the set of vertices incident with an edge in $A$. Thus there are two natural polymatroids one can define on the edges of a graph, $h_G$ and $M_G$ (the cycle matroid of $G$). Of course, the two are not unrelated: essentially $M_G$ is the Dilworth truncation of $h_G$ in the sense of [6] (see also [9]). Given this connection, one might hope for a connection to exist between the excluded minors for the classes of Boolean 2-polymatroids and graphic matroids.

It is well known that the excluded minors for graphic matroids are $M^*(K_5)$, $M^*(K_{3,3})$, $PG(2,2)$, $PG^*(2,2)$, and $U_{2,4}$. The only connection between this set and the set of excluded minors for Boolean 2-polymatroids is that $U_{2,4}$ is the Dilworth truncation of $h_8$; the others are unrelated. The reason for this is that, while the Dilworth truncation of a Boolean 2-polymatroid is always a graphic matroid, there are many non-Boolean 2-polymatroids whose Dilworth truncations are graphic matroids.

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